

# Five-dimensional vacuum Einstein spacetimes in C-metric like coordinates

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## Abstract

A 5-dimensional Einstein spacetime with (non)vanishing cosmological constant is analyzed in detail. The metric is in close analogy with the 4-dimensional massless uncharged C-metric in many aspects. The coordinate system, horizons and causal structures, relations to standard form of de Sitter, anti de Sitter and Minkowski vacua are investigated. After a boost and Kaluza-Klein reduction, we get an exact solution of 4-dimensional Einstein-Maxwell-Liouville theory which reduces to a solution to Einstein-Liouville theory in the limit of zero boost velocity and to that of Einstein-Maxwell-dilaton theory in the case of zero cosmological constant.

## 1 Introduction

The study of higher-dimensional gravity attracted considerable attentions in the recent years. The reason is two folded: one reason lies in that string theory requires higher spacetime dimensions, the other reason is that studies of gravity in higher dimensions might reveal deeper structures of general relativity, such as stability issues and classification of singularities of spacetime in various dimensions. For recent reviews in the subject, see [1] and [2].

In [3], two of the present authors obtained and analyzed in detail an exact 5-dimensional (5D) C-metric like vacuum solution of the Einstein equation with non-negative cosmological constant. The C-metric in 4D has been known for a long time [4]. The original C-metric and subsequent generalizations were shown to admit very interesting physical features [5, 6, 7, 8, 9]. Despite the intensive studies on higher dimensional gravities, it is a bit strange that no higher dimensional

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analogue of C-metric has been found so far. It was concluded very recently [10][11] that there are no generalizations of the C-metric in the Robinson-Trautman family with black hole horizons in higher dimensions. The spacetime we obtained in [3] does not contain any black hole horizons, but it does have two acceleration horizons and interestingly nontrivial global structure. We also interpreted the special case of  $\Lambda = 0$  using exterior geometric technique following [12].

In the discussion section of [3], another C-metric like vacuum solution was presented but not analyzed in detail. In this paper, we shall be dealing with this novel metric with arbitrary value of  $\Lambda$ . The plan of this paper is as follows. In section 2 we give the metric and its coordinate ranges. In section 3 we study the causal structures of the metric in various cases. In section 4 we analyze the accelerating nature of the horizons. Section 5 is devoted to the connection to the standard dS, AdS and Minkowski spacetimes using exterior geometric techniques. Then, in section 6, we give a 4D interpretation of the metric via Kaluza-Klein reduction, which leads to an exact solution for Einstein-Maxwell-Liouville theory just as the metric analyzed in [3]. Finally, in section 7, some concluding remarks are presented.

## 2 The metric and coordinate ranges

The metric to be studied is

$$ds^2 = \frac{1}{\alpha^2(x+y)^2} \left[ -G(y)dt^2 + \frac{dy^2}{G(y)} + \frac{dx^2}{F(x)} + F(x) \left( \frac{dz^2}{H(z)} + H(z)d\phi^2 \right) \right], \quad (1)$$

where

$$F(x) = 1 - x^2, \quad G(y) = -1 - \frac{\Lambda}{6\alpha^2} + y^2, \quad H(z) = 1 - z^2. \quad (2)$$

By straightforward calculations it can be seen that this metric is an exact solution of the 5D vacuum Einstein equation  $R_{MN} - \frac{1}{2}g_{MN}R + \Lambda g_{MN} = 0$ . So the corresponding spacetime is a 5D vacuum Einstein spacetime for every constant value of the cosmological constant  $\Lambda$ .

Obviously, the function  $F(x)$  and  $H(z)$  both have two zeros

$$x = \pm 1, \quad z = \pm 1$$

which constrain the range of values for the coordinates  $x$  and  $z$ . The zeros for  $G(y)$  depends on the value of the cosmological constant  $\Lambda$ , which can be divided into the following cases:

1. for  $\Lambda > 0$  (dS case),  $G(y)$  has two zeros

$$y = \pm y_0, \quad y_0 = \sqrt{1 + \frac{\Lambda}{6\alpha^2}} > 1;$$

2. for  $\Lambda = 0$  (Minkowski case),  $G(y)$  has two zeros  $y = \pm 1$ ;
3. for  $\Lambda < 0$  (AdS case), the situation is a little more complicated:
  - if  $-6\alpha^2 < \Lambda < 0$ ,  $G(y)$  has two zeros

$$y = \pm y_0, \quad y_0 = \sqrt{1 + \frac{\Lambda}{6\alpha^2}} < 1;$$

- if  $\Lambda = -6\alpha^2$ ,  $G(y)$  has a double zero  $y = 0$ ;
- if  $\Lambda < -6\alpha^2$ ,  $G(y)$  has no zeros.

Since  $y$  plays the role of radial coordinate, its range is not restricted by the zeros of  $G(y)$ . However, the overall conformal factor in the metric implies that  $x + y = 0$  is the conformal infinity, and in general it suffices to consider the spacetime located on a single side of the conformal infinity. Without loss of generality, we make the choice  $x + y \geq 0$ . Thus we get a variable range of  $y$  depending on the value of  $x$ ,

$$y \in [-x, \infty).$$

Putting together, the coordinate ranges of the metric (1) are given as follows,

$$\begin{aligned} t &\in (-\infty, \infty), & y &\in [-x, \infty), & x &\in [-1, 1], \\ z &\in [-1, 1], & \phi &\in [0, 2\pi). \end{aligned}$$

What is the role of zeros of the function  $G(y)$  listed above? It will be shown later that many but not all of these zeros correspond to acceleration horizons. A given zero  $y'$  (which can take either one of the values  $\pm y_0$ ) of  $G(y)$  corresponds to a horizon if and only if it is located inside the physical range  $[-x, \infty)$  of  $y$ , i.e. it satisfies the inequality  $x + y' > 0$ . All  $y$ 's whose values violate the above inequality are not horizons.

## 3 Causal structures

### 3.1 $\Lambda > 0$

The procedure to draw the Carter-Penrose diagram is as follows. First, for convenience, we change the coordinates  $x$  and  $z$  into  $\theta_1$  and  $\theta_2$  via

$$\theta_1 = \arccos(-x), \quad \theta_2 = \arccos(z)$$

so that the angular part of the metric looks like that of a 3-sphere,

$$d\Omega_3^2 = d\theta_1^2 + \sin^2 \theta_1 (d\theta_2^2 + \sin^2 \theta_2 d\phi^2). \quad (3)$$

Then we introduce the Eddington-Finkelstein coordinates,

$$u = t - y^*, \quad v = t + y^*, \quad (4)$$

where the tortoise coordinate  $y^*$  is defined as

$$y^* = \int G^{-1} dy = \frac{1}{2y_0} \log \left| \frac{y - y_0}{y + y_0} \right|, \quad (5)$$

and both  $u$  and  $v$  belong to the range  $(-\infty, \infty)$ . In this coordinate the metric becomes

$$ds^2 = \frac{r^2}{\alpha^2} [-G(y) du dv + d\Omega_3^2], \quad r = (x + y)^{-1}. \quad (6)$$

The Kruskal coordinates are introduced as

$$\tilde{u} = \pm \exp(-y_0 u), \quad \tilde{v} = \pm \exp(y_0 v),$$

where  $\tilde{u}$  and  $\tilde{v}$  takes the same sign if  $-x < y < y_0$ , and they take opposite signs if  $y \geq y_0$ . So there are totally 4 different combinations, each of which corresponds to a causal patch in the conformal diagrams to be drawn below. In each cases, one finds that

$$\tilde{u}\tilde{v} = -\frac{y - y_0}{y + y_0}, \quad (7)$$

and eq.(6) becomes

$$ds^2 = \frac{r^2}{\alpha^2} \left[ -\frac{(y + y_0)^2}{y_0^2} d\tilde{u} d\tilde{v} + d\Omega_3^2 \right], \quad (8)$$

where  $y$  and  $r$  are to be regarded as functions of  $\tilde{u}$  and  $\tilde{v}$ ,

$$\begin{aligned} y &= y_0 \frac{1 - \tilde{u}\tilde{v}}{1 + \tilde{u}\tilde{v}}, \\ r &= \frac{1 + \tilde{u}\tilde{v}}{(y_0 + x) - \tilde{u}\tilde{v}(y_0 - x)}. \end{aligned}$$

Finally, the Carter-Penrose coordinates can be introduced by the usual arctangent mappings of  $\tilde{u}$  and  $\tilde{v}$

$$\begin{aligned} U &= \arctan \tilde{u}, \quad V = \arctan \tilde{v}, \\ T &\equiv U + V, \quad R \equiv U - V, \end{aligned}$$

in terms of which the metric becomes

$$ds^2 = \frac{1}{\alpha^2 (y_0 \cos T - \cos \theta_1 \cos R)^2} [-dT^2 + dR^2 + \cos^2(R) d\Omega_3^2]. \quad (9)$$

The values of the product  $\tilde{u}\tilde{v}$  at  $y = y_0$ ,  $r = 0$  and  $r = \infty$  are respectively

$$\lim_{y \rightarrow y_0} \tilde{u}\tilde{v} = 0, \quad \lim_{r \rightarrow 0} \tilde{u}\tilde{v} = -1, \quad \lim_{r \rightarrow \infty} \tilde{u}\tilde{v} = \frac{y_0 + x}{y_0 - x}. \quad (10)$$

These correspond to the various causal boundaries in the Carter-Penrose diagrams. From the coordinate transformations introduced above, it is easy to see that if the limit of  $\tilde{u}\tilde{v}$  is 0 or  $\infty$ , the corresponding line is mapped into a null line; if the limit of  $\tilde{u}\tilde{v}$  is  $-1$ , the corresponding line is mapped into a spacelike line; if the limit is 1, the corresponding line is mapped into a timelike line. Since for  $\Lambda > 0$  we have  $y_0 > 1$ ,  $\lim_{r \rightarrow \infty} \tilde{u}\tilde{v}$  has a finite positive value varying with  $x$ . Therefore, the Carter-Penrose diagram in this case consists of the coordinate poles represented by the lines  $r = 0$ , the horizon at  $y = y_0$  and the (curved) past and future conformal infinities at  $r = \infty$ . The corresponding diagram is depicted as in Fig.1 (a). In this and all subsequent figures, lines labeled with  $y = y_0$  or  $y = -y_0$  represent acceleration horizons,  $I^+$  and  $I^-$  are respectively past and future infinities ( $r = +\infty$  or  $y = -x$ ), and  $r = 0$  or  $I$  correspond to  $y = +\infty$ , the spacelike infinities.

Notice that the conformal infinities become exactly timelike (i.e. straight horizontal lines) at the particular value  $x = 0$ . A similar Carter-Penrose diagram has been found for the 4D massless uncharged de Sitter C-metric in [9].

### 3.2 $\Lambda = 0$

The procedure for drawing Carter-Penrose diagrams for the case  $\Lambda = 0$  is very similar to the case  $\Lambda > 0$ , the only difference lies in that we need to set  $y_0$  to the specific value 1, and under this value of  $y_0$  the analysis of  $\lim_{r \rightarrow \infty} \tilde{u}\tilde{v}$  is a little bit more complicated:

- $-1 < x < 1$  :  $y = 1$  can be reached and is a horizon.  $\lim_{r \rightarrow \infty} \tilde{u}\tilde{v}$  is again finite, positive and varying with  $x$ . The Carter-Penrose diagram is the same as that for the  $\Lambda > 0$  case, i.e. Fig.1 (a);
- $x = 1$  : both the horizon at  $y = 1$  and the boundary at  $y = -1$  can be reached, with the latter being null infinities. The Carter-Penrose diagram is depicted in Fig.1 (b);
- $x = -1$  : the minimum value of  $y$  is  $y = 1$ , which overlaps with null conformal infinities. The Carter-Penrose diagram is depicted in Fig.1 (c).

The same diagrams have appeared and were combined into the global visualization in [6] which correspond to the massless uncharged C-metric without cosmological constant in 4D.

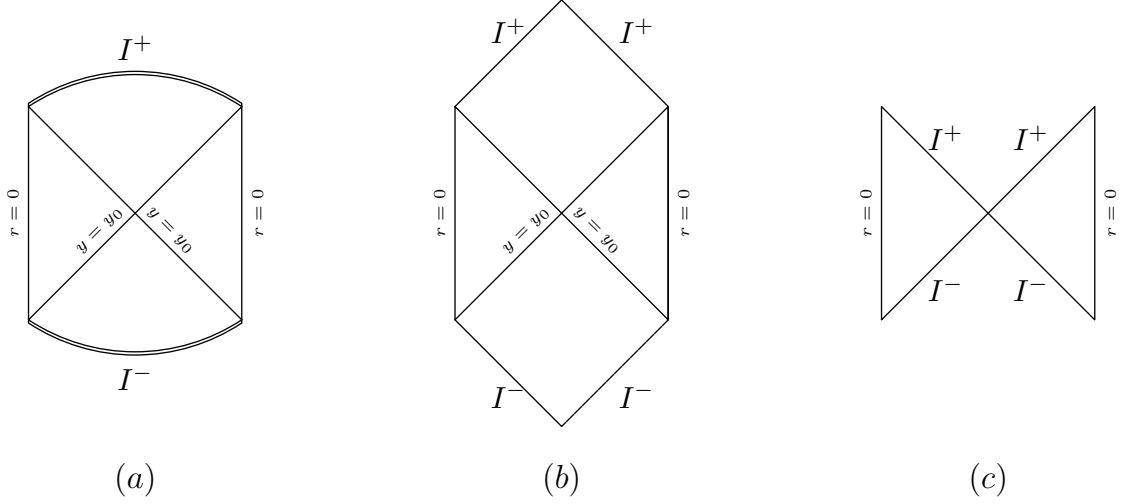


Figure 1: Carter-Penrose diagrams for  $\Lambda \geq 0$ : (a) corresponds to both  $\Lambda > 0$  and the  $-1 < x < 1$  case of  $\Lambda = 0$ ; (b) and (c) respectively corresponds to  $\Lambda = 0$  with  $x = 1$  and  $x = -1$ .

### 3.3 $\Lambda < 0$

For  $\Lambda < 0$  we have to analyze separately three subcases  $-6\alpha^2 < \Lambda < 0$ ,  $\Lambda = -6\alpha^2$  and  $\Lambda < -6\alpha^2$ . Since the lower bound for the coordinate  $y$  depends on  $x$ , the Carter-Penrose diagrams will also change according to the value of  $x$ .

#### 3.3.1 $-6\alpha^2 < \Lambda < 0$

This case corresponds to  $0 < y_0 < 1$  and the procedure to get the final conformal diagrams is basically the same as before. In the final step for analysing  $\lim_{r \rightarrow \infty} \tilde{u}\tilde{v}$ , we need to subdivide the range of  $x$  into five subcases:  $-1 \leq x < -y_0$ ,  $x = -y_0$ ,  $-y_0 < x < y_0$ ,  $x = y_0$  and  $y_0 < x < 1$ .

- $-1 \leq x < -y_0$ : in this case  $\lim_{r \rightarrow \infty} \tilde{u}\tilde{v}$  takes a finite negative value which is varying with  $x$ . There is no horizons because  $y_0$  is beyond the physical region  $[-x, \infty)$  for  $y$ . The corresponding causal diagram is depicted in Fig.2 (a).
- $x = -y_0$ :  $y = y_0$  overlaps with the conformal infinities at  $r = \infty$ , so there is no horizon and the Carter-Penrose diagram is shown in Fig.2 (b).
- $-y_0 < x < y_0$ :  $y = y_0$  is the only horizon because  $-y_0$  is beyond the physical region of the coordinate  $y$ . The Carter-Penrose diagram is similar to the  $\Lambda > 0$  case and is depicted in Fig.2 (c).
- $x = y_0$ :  $y = y_0$  is the only horizon and  $y = -y_0$  overlaps with the null conformal infinities. The Carter-Penrose diagram is shown in Fig.2 (d).

- $y_0 < x < 1$  :  $y = y_0$  and  $y = -y_0$  are both horizons and  $\lim_{r \rightarrow \infty} \tilde{u}\tilde{v}$  takes a finite negative value which is varying with  $x$ . The Carter-Penrose diagram is shown in Fig.2 (e). Note that for this case, the diagram is vertically not bounded – there are repeating copies in the time direction.

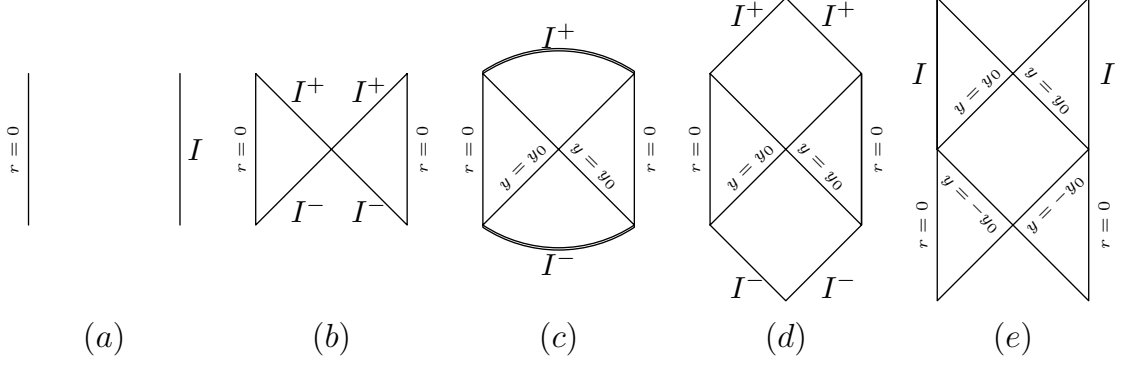


Figure 2: Carter-Penrose diagrams for  $-6\alpha^2 < \Lambda < 0$

### 3.3.2 $\Lambda = -6\alpha^2$

This case corresponds to  $y_0 = 0$  and  $G(y) = y^2$ . The tortoise coordinates given in (5) now cease to be meaningful, and we need to introduce  $y^*$  in a  $y_0$ -independent way. Actually, inserting  $G(y) = y^2$  into the first equality of (5) naturally gives the correct  $y^*$  for the present case,

$$y^* = \int \frac{dy}{y^2} = -\frac{1}{y}.$$

So, introducing the sequence of coordinate changes

$$\begin{aligned} u &= t - y^*, & v &= t + y^*, \\ U &= \arctan u, & V &= \arctan v, \\ T &= U + V, & R &= U - V, \end{aligned}$$

the metric can be rewritten as

$$ds^2 = \frac{r^2}{\alpha^2 \sin^2 R} (-dT^2 + dR^2 + \sin^2 R d\Omega_3^2). \quad (11)$$

We need to subdivide the values of  $x$  into 3 regions:

- $-1 < x < 0$  : No horizons exist and the Carter-Penrose diagram is depicted as in Fig.3 (a);

- $x = 0 : y_0 = 0$  overlaps with the conformal infinities and hence no horizons exist and the Carter-Penrose diagram is depicted as in Fig.3 (b).
- $0 < x < 1$  : There is a double horizon at  $y = 0$  and the Carter-Penrose diagram is depicted as in Fig.3 (c).

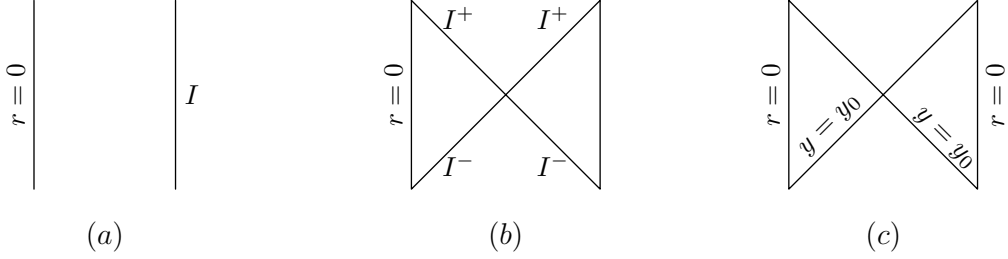


Figure 3: Carter-Penrose diagrams for  $\Lambda = -6\alpha^2$

### 3.3.3 $\Lambda < -6\alpha^2$

In this case we write  $G(y) = y^2 + y_0^2$ , with  $y_0^2 = -1 - \frac{\Lambda}{6\alpha^2} > 0$ . Then the tortoise coordinate can be introduced as

$$y^* = \int G^{-1}(y) dy = \frac{1}{y_0} \arctan \frac{y}{y_0}.$$

Introducing the new coordinates  $T = y_0 t$ ,  $R = y_0 y^*$ , the metric can be rewritten as

$$ds^2 = \frac{r^2}{\alpha^2 \cos^2 R} (-dT^2 + dR^2 + \cos^2 R d\Omega_3^2).$$

Contrary to the cases  $-6\alpha^2 < \Lambda < 0$  and  $\Lambda = -6\alpha^2$ , the Carter-Penrose diagram of this case does not depend on the value of  $x$  and contains no horizons. The corresponding diagram is depicted as in Fig.3 (a).

In closing this subsection, let us remark that for each of the Carter-Penrose diagrams corresponding to  $\Lambda < 0$  listed above, one can find a 4D counter part for it in [8] respectively<sup>1</sup>. Combining with the results obtained in previous subsections, we conclude that the Carter-Penrose diagrams for our metric with any value of  $\Lambda$  are all analogous to the corresponding diagrams for the 4D massless uncharged C-metrics. The only difference is that the geometry of each point in the diagrams are different (with one extra angular dimension in our case). However, the same shapes of the Carter-Penrose diagrams imply that our metric (1) is the 5D generalization of the massless uncharged 4D C-metrics.

<sup>1</sup>For 4D massive C-metric with negative cosmological constant, the causal structure was also studied by Krtous in [13], where the conformal diagrams are patched together to yield 3D global view.



## 4 Acceleration horizons

In this section we shall show that every horizon appeared in Section 3 is an acceleration horizon. This is achieved by use of some properly chosen coordinate transformations. In this process, the meaning of the parameter  $\alpha$  will also get clear: it is just the magnitude of the acceleration of the origin of the coordinate systems which we are going to choose.

### 4.1 $\Lambda > 0$

Let us first perform the following coordinate transformation [14]

$$\begin{aligned}\tau &= \alpha^{-1}\kappa t, & \rho &= \alpha^{-1}\kappa y^{-1}, \\ \theta_1 &= \arccos(-x), & \theta_2 &= \arccos z,\end{aligned}\tag{12}$$

with

$$\ell^2 = 6/\Lambda, \quad \kappa = \sqrt{1 + \alpha^2 \ell^2}.\tag{13}$$

Doing so the metric (1) becomes

$$ds^2 = \frac{1}{\gamma^2} \left[ -(1 - \rho^2/\ell^2) d\tau^2 + \frac{d\rho^2}{1 - \rho^2/\ell^2} + \rho^2 d\Omega_3^2 \right],\tag{14}$$

where

$$\gamma = \kappa - \alpha \rho \cos \theta_1.\tag{15}$$

Now consider the timelike observer at fixed spacial position  $(\rho, \theta_1, \theta_2, \phi)$  in the spacetime described by the worldline  $x^\mu(\lambda) = (\gamma \ell \lambda / \sqrt{\ell^2 - \rho^2}, \rho, \theta_1, \theta_2, \phi)$ , where  $\lambda$  is the proper time. The acceleration of the observer,  $a^\mu = u^\nu \nabla_\nu u^\mu$ , where  $u^\mu = dx^\mu/d\lambda$  is the proper velocity obeying  $u^\mu u_\mu = -1$ , has a magnitude

$$a^\mu a_\mu = \alpha^2 + \frac{1}{\ell^2(\ell^2 - \rho^2)} (\kappa^2 \rho^2 - 2\ell^2 \alpha \kappa \rho \cos \theta_1 + \ell^2 \alpha^2 \rho^2 \cos^2 \theta_1).\tag{16}$$

We see that the observer at the origin  $\rho = 0$  (or  $y = \infty$ ), is being accelerated with a constant acceleration  $|a| = \alpha$ . Notice that the choice  $\alpha = 0$  makes the metric (14) become that of the usual dS spacetime in static spherical coordinates. Moreover, at  $\rho = \ell$  (or  $y = y_0$ ), the acceleration becomes infinite which corresponds to the trajectory of a null ray. All observers held at  $\rho = \text{const}$  see the null ray as an acceleration horizon and they will never see events beyond this null ray.

## 4.2 $\Lambda = 0$

For  $\Lambda = 0$  we perform the following coordinate transformation

$$\begin{aligned}\tau &= t, & \rho &= y^{-1}, \\ \theta_1 &= \arccos(-x), & \theta_2 &= \arccos z.\end{aligned}\tag{17}$$

The metric then becomes

$$ds^2 = \frac{1}{\gamma^2} \left[ -(1 - \rho^2) d\tau^2 + \frac{d\rho^2}{1 - \rho^2} + \rho^2 d\Omega_3^2 \right],\tag{18}$$

where

$$\gamma = \alpha(1 - \rho \cos \theta_1).\tag{19}$$

Consider the timelike observer at fixed spacial position described by the worldline  $x^\mu(\lambda) = (\gamma\lambda/\sqrt{1 - \rho^2}, \rho, \theta_1, \theta_2, \phi)$ . The proper acceleration  $a^\mu = (\nabla_\nu u^\mu)u^\nu$  now has a magnitude

$$a^\mu a_\mu = \frac{\alpha^2}{1 - \rho^2} (1 - \rho \cos \theta_1)^2.$$

The observer at  $\rho = 0$  (or  $y = \infty$ ) is being accelerated with a constant acceleration  $|a| = \alpha$ , while those at  $\rho = 1$  (or  $y = y_0 = 1$ ) are being accelerated with infinite acceleration. This shows that for  $\Lambda = 0$ ,  $y = y_0 = 1$  is indeed an accelerating horizon.

## 4.3 $\Lambda < 0$

We need to consider 3 different cases.

### 4.3.1 $-6\alpha^2 < \Lambda < 0$

After performing the coordinate transformation of the same form with (12) but with

$$\ell^2 = -6/\Lambda, \quad \kappa = \sqrt{\alpha^2 \ell^2 - 1},\tag{20}$$

we can rewrite the metric (1) in the form (14) with  $\gamma$  taking the form (15). Then we consider the timelike observer  $x^\mu(\lambda) = (\gamma\ell\lambda/\sqrt{\ell^2 - \rho^2}, \rho, \theta_1, \theta_2, \phi)$  at fixed spacial position. The magnitude of the proper acceleration observed by this observer takes exactly the same form as (16), now with  $\ell$  and  $\kappa$  given by (20). The observer at  $\rho = 0$  (or  $y = \infty$ ) is accelerated with a constant acceleration  $|a| = \alpha$ , and at  $\rho = \ell$  (or  $y = y_0$ ), the acceleration becomes infinite, signifying the existence of an acceleration horizon.

#### 4.3.2 $\Lambda = -6\alpha^2$

In this case we have  $G(y) = y^2$ , i.e.  $y_0 = 0$ . A possible coordinate change is given by

$$\begin{aligned}\tau &= \alpha^{-1}t, & \rho &= \alpha^{-1}y^{-1}, \\ \theta_1 &= \arccos(-x), & \theta_2 &= \arccos z,\end{aligned}\tag{21}$$

after which the metric becomes

$$ds^2 = \frac{1}{\gamma^2} (-d\tau^2 + d\rho^2 + \rho^2 d\Omega_3^2)\tag{22}$$

with

$$\gamma = 1 - \alpha\rho \cos \theta_1.\tag{23}$$

The metric (22) does not explicitly contain a horizon, and so is not applicable for evaluating infinite magnitude of the proper acceleration at the horizon. However, this form of the metric is explicitly conformal to Minkowski metric, as apposed to the previous 3 cases which are conformal to the standard form of de Sitter metric, so we keep it here.

To actually realize that the double horizon at  $y_0 = 0$  is indeed an acceleration horizon, we take an alternative route. We observe that the same metric (14) can also describe the Einstein spacetime with  $\Lambda = -6\alpha^2$ , provided  $\gamma$  is given in the form (15) with

$$\kappa = \sqrt{2}\alpha\ell.\tag{24}$$

Therefore, we can borrow the result from the previous case (i.e. the case  $-6\alpha^2 < \Lambda < 0$ ) and taking the limit  $\kappa \rightarrow \sqrt{2}\alpha\ell$ . In this way we should again find that the timelike observer at fixed spacial position will observe a proper acceleration of magnitude  $a^\mu a_\mu = \infty$  at the horizon  $\rho = \ell$  and  $a^\mu a_\mu = \alpha^2$  at  $\rho = 0$ .

#### 4.3.3 $\Lambda < -6\alpha^2$

Now we make the coordinate transformation of the same form with (12) but with

$$\ell^2 = -6/\Lambda, \quad \kappa = \sqrt{1 - \alpha^2\ell^2}.\tag{25}$$

Doing so the metric (1) is turned into the form

$$ds^2 = \frac{1}{\gamma^2} \left[ -(1 + \rho^2/\ell^2)d\tau^2 + \frac{d\rho^2}{1 + \rho^2/\ell^2} + \rho^2 d\Omega_3^2 \right]\tag{26}$$

with  $\gamma$  taking the form (15). Then the magnitude of the acceleration for the timelike observer  $x^\mu(\lambda) = (\gamma\ell\lambda/\sqrt{\ell^2 + \rho^2}, \rho, \theta_1, \theta_2, \phi)$  at fixed spacial position turns out to be

$$a^\mu a_\mu = \alpha^2 + \frac{1}{\ell^2(\ell^2 + \rho^2)} (\kappa^2 \rho^2 + 2\ell^2 \alpha \kappa \rho \cos \theta_1 - \ell^2 \alpha^2 \rho^2 \cos^2 \theta_1).$$

At  $\rho = 0$  (or  $y = \infty$ ) the acceleration is again the same constant  $|a| = \alpha$ . However, the magnitude of the acceleration never blows up to infinity, showing that there is no acceleration horizons, which is in agreement with the fact that  $G(y)$  has no zeros in the original metric in this case.

#### 4.4 Area of the horizons

Having established that real zeros of  $G(y)$  correspond to acceleration horizons, we now calculate the area of these horizons. The metric of the horizons can be written as

$$ds_H^2 = \frac{\ell^2}{\gamma^2} d\Omega_3^2,$$

where  $\gamma$  is given by (15) or (19) with  $\rho = \ell$  according to different values of  $\Lambda$ .

The area of the horizons can be easily evaluated using the formula

$$A = \int d\theta_1 d\theta_2 d\phi \sqrt{g_H},$$

where the integration with respect to  $\phi$  and  $\theta_2$  is taken over  $\phi \in [0, 2\pi)$ ,  $\theta_2 \in [0, \pi)$ . The integration with respect to  $\theta_1$  depends on the value of  $\Lambda$ : for  $\Lambda \geq 0$ , the integration is taken over  $\theta_1 \in [0, \pi)$ , while for  $-6\alpha^2 \leq \Lambda < 0$ , the integration is taken over  $\theta_1 \in [0, \pi/2)$ .

For  $\Lambda > 0$ , we get a finite result

$$A = 2\pi^2 \left( \frac{6}{\Lambda} \right)^{3/2}.$$

For  $-6\alpha^2 \leq \Lambda \leq 0$  cases, the area of the horizons always diverges. This result is in agreement with the common knowledge that the Einstein manifolds with  $\Lambda > 0$  are compact, while those with  $\Lambda \leq 0$  are non-compact.

### 5 Exterior geometry and relation to dS, AdS and Minkowski spacetimes

By direct calculations it can be seen that the Weyl tensor for metric (1) is identically zero. This signifies that the spacetime is conformally flat, very similar to the case of the standard form of dS, AdS and Minkowski spacetimes. In this section, we are aimed at exploring the relationship between our metric (1) and the standard dS/AdS/ Minkowski spacetimes. It will be shown that our metric is indeed equivalent to the standard dS/AdS/Minkowski spacetimes, although written in different coordinate systems. For this reason we adopt some exterior geometric techniques and consider the  $\Lambda > 0$  and  $\Lambda < 0$  cases as some hyperboloids embedded in 6D flat

spacetime. For  $\Lambda = 0$ , however, we shall show directly that our metric is the 5D Minkowski spacetime written in a particular coordinate system.

### 5.1 $\Lambda > 0$

The de Sitter spacetime can be represented as the 5-hyperboloid

$$-(X_0)^2 + (X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 + (X_5)^2 = \ell^2 \quad (27)$$

embedded in the 6D Minkowski spacetime

$$ds^2 = -(dX_0)^2 + (dX_1)^2 + (dX_2)^2 + (dX_3)^2 + (dX_4)^2 + (dX_5)^2 \quad (28)$$

with  $\ell^2 = 6/\Lambda$  and  $\Lambda > 0$ .

Our metric for  $\Lambda > 0$ , presented in the form (14), fits nicely in the above 6D picture if we parametrize the 5D hyperboloid (27) with the following coordinate transformation

$$\begin{aligned} X_0 &= \gamma^{-1} \sqrt{\ell^2 - \rho^2} \sinh(\tau/\ell), & X_2 &= \gamma^{-1} \rho \sin \theta_1 \sin \theta_2 \sin \phi, \\ X_1 &= \gamma^{-1} \sqrt{\ell^2 - \rho^2} \cosh(\tau/\ell), & X_3 &= \gamma^{-1} \rho \sin \theta_1 \sin \theta_2 \cos \phi, \\ X_5 &= \gamma^{-1} [-\kappa \rho \cos \theta_1 + \alpha \ell^2], & X_4 &= \gamma^{-1} \rho \sin \theta_1 \cos \theta_2, \end{aligned}$$

where  $\rho$ ,  $\tau$  and  $\gamma$  given by (12) and (15) with  $\ell$  and  $\kappa$  given by (13). Similar transformations can be found in [14], while treating the massless uncharged limit of the 4D dS C-metric. This justifies that the metric (14) (and hence (1) for  $\Lambda > 0$ ) is just the 5D de Sitter spacetime in a particular accelerating coordinate system.

### 5.2 $\Lambda = 0$

We will show that for this particular value of  $\Lambda$ , our metric is just the 5D Minkowski spacetime written in a particular coordinate system. For this purpose, we will not resort to embedding into 6D spacetime but stick to 5D description instead. The procedure to be carried out below is inspired by the work of [12] and the logic is basically the same as in the  $\Lambda = 0$  case of [3].

Consider the following 4D algebraic surface embedded in a 5D Euclidean space,

$$X_1^2 + X_2^2 + X_3^2 + \left( \sqrt{X_4^2 + X_5^2} - a \right)^2 = b^2. \quad (29)$$

For constants  $a > b$ , this equation describes a compact 4-dimensional surface of topology  $S^3 \times S^1$ . In fact, the surface can be thought of as the result of pulling the center of a 3-sphere of radius  $b$  everywhere around a circle of radius  $a$  lying in different dimensions.

We can parametrize the above surface in 5D Euclidean space as follows:

$$\begin{aligned} X_1 &= \frac{\alpha}{B} \sin \theta \sin \chi \cos \phi, & X_2 &= \frac{\alpha}{B} \sin \theta \sin \chi \sin \phi, & X_3 &= \frac{\alpha}{B} \sin \theta \cos \chi, \\ X_4 &= \frac{\alpha}{B} \sinh \eta \cos \psi, & X_5 &= \frac{\alpha}{B} \sinh \eta \sin \psi, \end{aligned} \quad (30)$$

where

$$\begin{aligned} B &\equiv \cosh \eta - \cos \theta, \\ \alpha &\equiv \sqrt{a^2 - b^2}, \end{aligned}$$

provided  $\eta$  takes the special value

$$\eta = \eta_0, \quad \cosh \eta_0 = \frac{a}{b}.$$

For variable values of  $\eta$ , (30) is just another parametrization of the 5D Euclidean space.

Making some further coordinate transform

$$x = -\cos \theta, \quad y = \cosh \eta, \quad z = \cos \chi,$$

it can be checked that the 5-dimensional Euclidean metric

$$ds^2 = \sum_{i=1}^5 dX_i^2$$

is equivalent to the Wick rotated version of the metric (1) with  $\Lambda = 0$ , i.e.

$$\begin{aligned} ds^2 &= \frac{1}{\alpha^2(x+y)^2} \left[ (y^2 - 1)d\psi^2 + \frac{dy^2}{y^2 - 1} \right. \\ &\quad \left. + \frac{dx^2}{1 - x^2} + (1 - x^2) \left( \frac{dz^2}{1 - z^2} + (1 - z^2)d\phi^2 \right) \right]. \end{aligned} \quad (31)$$

To actually obtain (1), we need to make a Wick rotation  $\psi \rightarrow it$ , which is equivalent to Wick rotating  $X_5$  in the above. This amounts to changing the geometry of the constant  $y$  slices of the spacetime to the embedding equation

$$X_1^2 + X_2^2 + X_3^2 + \left( \sqrt{X_4^2 - X_5^2} - a \right)^2 = b^2$$

in a 5D Minkowski spacetime. The above equation simply describes the result of pulling the center of a 3-sphere everywhere along a pair of hyperbolas.

### 5.3 $\Lambda < 0$

The AdS spacetime can be represented as the 5-hyperboloid

$$-(X_0)^2 + (X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 - (X_5)^2 = -\ell^2 \quad (32)$$

embedded in the 6D spacetime

$$ds^2 = -(dX_0)^2 + (dX_1)^2 + (dX_2)^2 + (dX_3)^2 + (dX_4)^2 - (dX_5)^2 \quad (33)$$

with  $\ell^2 = -6/\Lambda$  and  $\Lambda < 0$ . For different values  $-6\alpha^2 < \Lambda < 0$ ,  $\Lambda = -6\alpha^2$  and  $\Lambda < -6\alpha^2$ , we need different parametrizations for (32).

#### 5.3.1 $-6\alpha^2 < \Lambda < 0$

In this case, our metric can be written as in (14). This metric follows from (32) and (33) if we parametrize (32) with the following coordinate transformation

$$\begin{aligned} X_0 &= \gamma^{-1} \sqrt{\ell^2 - \rho^2} \sinh(\tau/\ell), & X_2 &= \gamma^{-1} \rho \sin \theta_1 \sin \theta_2 \sin \phi, \\ X_1 &= \gamma^{-1} \sqrt{\ell^2 - \rho^2} \cosh(\tau/\ell), & X_3 &= \gamma^{-1} \rho \sin \theta_1 \sin \theta_2 \cos \phi, \\ X_5 &= \gamma^{-1} [-\kappa \rho \cos \theta_1 + \alpha \ell^2], & X_4 &= \gamma^{-1} \rho \sin \theta_1 \cos \theta_2. \end{aligned}$$

with  $\rho$ ,  $\tau$  and  $\gamma$  given by (12) and (15) with  $\ell$  and  $\kappa$  given in (20). Similar transformations can be found in [8], while treating the massless uncharged limit of the 4D AdS C-metric. This justifies that the metric (1) with  $-6\alpha^2 < \Lambda < 0$  is just the 5D AdS spacetime written in a particular accelerating coordinate system.

#### 5.3.2 $\Lambda = -6\alpha^2$

In this case, our metric can be written as in (22). This metric follows from (32) and (33) if we introduce the following coordinate transformation

$$\begin{aligned} X_0 &= \eta^{-1} \tau, & X_2 &= \eta^{-1} \rho \sin \theta_1 \sin \theta_2 \sin \phi, \\ X_1 &= \frac{1}{2} \eta^{-1} [1 - (\rho^2 \sin^2 \theta_1 + \ell^2 \eta^2 - \tau^2)], & X_3 &= \eta^{-1} \rho \sin \theta_1 \sin \theta_2 \cos \phi, \\ X_5 &= \frac{1}{2} \eta^{-1} [1 + (\rho^2 \sin^2 \theta_1 + \ell^2 \eta^2 - \tau^2)], & X_4 &= \eta^{-1} \rho \sin \theta_1 \cos \theta_2, \end{aligned}$$

where  $\eta = \gamma/\alpha$  and  $\gamma$  is given by (23).

#### 5.3.3 $\Lambda < -6\alpha^2$

In this case, our metric takes the form (26), and it also follows from (32) and (33) if we parametrize (32) with the following coordinate transformation

$$X_0 = \gamma^{-1} \sqrt{\ell^2 + \rho^2} \sinh(\tau/\ell), \quad X_2 = \gamma^{-1} \rho \sin \theta_1 \sin \theta_2 \sin \phi,$$

$$\begin{aligned} X_5 &= \gamma^{-1} \sqrt{\ell^2 + \rho^2} \cosh(\tau/\ell), & X_3 &= \gamma^{-1} \rho \sin \theta_1 \sin \theta_2 \cos \phi, \\ X_1 &= \gamma^{-1} [-\kappa \rho \cos \theta_1 - \alpha \ell^2], & X_4 &= \gamma^{-1} \rho \sin \theta_1 \cos \theta_2 \end{aligned}$$

with  $\rho$ ,  $\tau$  and  $\gamma$  given by (12) and (15) with  $\ell$  and  $\kappa$  given in (25). Similar transformations can be found in [15], while treating the massless uncharged limit of the 4D AdS C-metric. This justifies that our metric (26) (hence (1) with  $\Lambda < -6\alpha^2$ ) is also an AdS spacetime written in a particular accelerating coordinate system.

Summarizing the above procedure, we conclude that the metric (1) is equivalent to either the standard dS or AdS spacetimes depending on the value of  $\Lambda$ . The same properties are shared by the weak field limit of the 4D C-metric as shown in detail in [8] and [9] respectively.

## 6 4D interpretation

The 5D metric (1) admits a nice 4D interpretation following the same fashion as we did in [3] for another metric. The basic strategy is to perform a boost and then make a Kaluza-Klein reduction.

The boost is now made in the  $t, \phi$  directions,

$$\begin{aligned} t &\rightarrow T = t \cosh \beta - \phi \sinh \beta, \\ \phi &\rightarrow \Phi = -t \sinh \beta + \phi \cosh \beta. \end{aligned}$$

Inserting the above into the metric (1), we get

$$\begin{aligned} d\tilde{s}_5^2 &= \frac{1}{\alpha^2(x+y)^2} \left[ -\frac{G(y) - k^2 F(x)H(z)}{1 - k^2} dT^2 + \frac{dy^2}{G(y)} + \frac{dx^2}{F(x)} + \frac{F(x)}{H(z)} dz^2 \right. \\ &\quad \left. + \frac{F(x)H(z) - k^2 G(y)}{1 - k^2} d\Phi^2 + \frac{2k(F(x)H(z) - G(y))}{1 - k^2} dT d\Phi \right], \end{aligned}$$

where the boost velocity  $k = \tanh \beta$  is used.

Now making a KK reduction along the  $\Phi$  axis using the formula

$$d\tilde{s}_5^2 = e^{\varphi/\sqrt{3}} d\tilde{s}_4^2 + e^{-2\varphi/\sqrt{3}} (d\Phi + \mathcal{A})^2.$$

we get the reduced 4D metric

$$\begin{aligned} d\tilde{s}_4^2 &= \frac{1}{\alpha^3(x+y)^3} \left( \frac{F(x)H(z) - k^2 G(y)}{1 - k^2} \right)^{1/2} \\ &\quad \times \left[ -\frac{F(x)G(y)H(z)(1 - k^2)}{F(x)H(z) - k^2 G(y)} dT^2 + \frac{dy^2}{G(y)} + \frac{dx^2}{F(x)} + F(x) d\theta_2^2 \right], \end{aligned}$$

where  $z = \cos \theta_2$ , together with the 4D Maxwell potential

$$\mathcal{A} = \frac{k[F(x)H(z) - G(y)]}{F(x)H(z) - k^2 G(y)} dT,$$



and the 4D Liouville field

$$e^{-2\varphi/\sqrt{3}} = \frac{1}{\alpha^2(x+y)^2} \frac{F(x)H(z) - k^2 G(y)}{1 - k^2}.$$

The above reduction can also be realized on the level of classical actions. Before the reduction, the action for the 5D vacuum Einstein equation  $R_{MN} - \frac{1}{2}g_{MN}R + \Lambda g_{MN} = 0$  is

$$S_5 = \int d^5x \sqrt{-g_{(5)}} (R_{(5)} - \Lambda), \quad (34)$$

up to a possible boundary counter term which we omit. After the KK reduction, the action becomes

$$S_4 = \int d^4x \sqrt{-g_{(4)}} \left( R_{(4)} - \frac{1}{2} (\partial\varphi)^2 - \Lambda e^{\varphi/\sqrt{3}} - \frac{1}{4} e^{\varphi/\sqrt{3}} F_{\mu\nu} F^{\mu\nu} \right),$$

where

$$F = F_{\mu\nu} dx^\mu \wedge dx^\nu \equiv d\mathcal{A}.$$

We see that this is the action of Einstein-Maxwell-Liouville theory. At  $k = 0$  the Maxwell field vanishes and only the 4D action reduces to that of the Einstein-Liouville theory. On the other hand, keeping  $k \neq 0$  while setting  $\Lambda = 0$ , we get a solution to the Einstein-Maxwell-dilaton theory.

## 7 Discussions

In this article, we analyzed the general properties of the metric (1) found in [3]. It turns out that this metric resembles very much to the 4D massless uncharged C-metric. In every cases of  $\Lambda > 0$ ,  $\Lambda = 0$  and  $\Lambda < 0$ , the Carter Penrose diagrams for the metric (1) were found to have the same shape as its 4D analogue found in [6], [9] and [8]. However, the appearance of a fifth dimension gives room for Kaluza-Klein reduction, which leads to a 4-dimensional interpretation in terms of Einstein-Maxwell-Liouville theory.

Potentially, the 5D metric studied here might be useful in finding 5D metrics with accelerating black holes inside and constructing black ring solutions in 6 dimensions. In the latter respect, the usual 4D C-metric has played a similar role in 5D black ring solutions [5]. Whether the same program can be carried on in the presence of one extra dimension remains to be investigated. We hope we can come back on that subject later on.

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